Independent learning task - Graph Theory

**Rationale**

Students of mathematics need to be good at reading a maths textbook and using it as a source for learning and revision. To help you develop these skills we would like you to complete this task before we look at the topic of Graph Theory. Don’t worry if you can’t do some of the questions as we will be covering the topic at the maths school at a future date, but you will learn much more if you are able to try and understand some of the concepts yourself before then. To successfully complete the task, you will need to be resourceful and resilient. If you get stuck on a question don’t let that put you off answering later questions, the question aren’t in order of difficulty and what you find easy or difficult will probably differ from others.

**What you need to do**

Work your way through each of the questions on the reverse of this sheet. This worksheet provides some definitions and examples of different graphs and asks you to get to grips with the basic ideas that underpin graph theory. Some of the questions will also guide you through some basic proofs.

Graph Theory is a very different type of mathematics to what you’ve seen so far at GCSE. Although it’s a very abstract part of mathematics, it has lots of real-world applications particularly when it comes to algorithms in computer science. At A level you’ll have the opportunity to study more graph theory in the Decision module of the course. Some of the ideas and proofs on this worksheet are outside the A level curriculum.

Graph Theory

A graph is defined as a set of vertices and a set of edges (). Vertices are points, and edges are lines (not necessarily straight) that connect two vertices together.

Chart, radar chart

Description automatically generatedFor example, the graph pictured below has:

* a vertex set: }. This is simply a list of all the vertices in the graph.
* an edge set: . This is a list of all the edges, where represents the line connecting and together. Note that we don’t include in the set as it’s the same edge as

A picture containing necklet

Description automatically generatedGraphs must also obey the following rules:

* Shape

  Description automatically generatedNo loops

*(A loop is when an edge begins and finishes on the same vertex.)*

* No multiple edges

Q1. a) Write down and for the graph shown.

Diagram

Description automatically generated

b) Draw the graph described by the following vertex and edge sets.

Chart

Description automatically generatedDefinition 1: The **degree,** or **order**, of a vertex is the number of edges incident to it. (A vertex and edge are incident if the vertex lies at either end of the edge).

has degree 1 as only one edge meets here.

has degree 3 as three edges meets here.

Definition 2: **Isomorphic graphs** are graphs with the same number of vertices connected in the same way with the same number of edges, but may be drawn differently.

Chart

Description automatically generated with low confidenceA picture containing diagram

Description automatically generatedQ2. The two graphs, and , below are isomorphic.

**:**

a) Copy and complete the table below. One row is already completed for you.

|  |  |  |
| --- | --- | --- |
| Vertex on | Corresponding Vertex on | Degree of vertex |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

b) Was there only one correct way to fill out the table above?

Definition 3: A **subgraph** of a graph , is a graph each of whose vertices and edges both belongs to . In other words, it is simply a part of .

Definition 4: A **finite graph** is a graph with a finite number of vertices.

A picture containing wire

Description automatically generated*Why do we define it as a finite number of vertices rather than a finite number of edges?*

Q3. The graph is drawn below.

1. Draw three non-isomorphic subgraphs of each with four vertices.
2. Identify which of graphs and are isomorphic to graph

Chart

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Chart

Description automatically generated with low confidenceChart

Description automatically generated

Q4. a) Prove that the sum of the degrees of the vertices of any finite graph is even.

*(Hint: Start with a graph containing vertices and no edges. What happens each time you add an edge to the graph?)*

b) Does this still hold if you allow graphs to have loops?

Definition 5: A **walk** is a route through a graph along edges, from one vertex to the next.

Definition 6: A **path** is a walk where no vertex is visited more than once.

Definition 7: A **trail** is a walk in which no edge is visited more than once.

We can use the notation to represent the path with vertices.

Chart, line chart

Description automatically generatedFor example:

A picture containing device, projector

Description automatically generated

A picture containing sky, wire

Description automatically generatedQ5. a) Draw the graph .

b) How many edges does the graph have?

c) What is the longest path, , you can find in the graph ? *List the vertices in the order you visit them.*

d) What is the longest trail you can find in the graph ? *List the vertices in the order you visit them.*

Definition 8: A **cycle** is a walk in which the end vertex is the same as the start vertex, but no other vertex is repeated more than once.

We can use the notation to represent the cycle with vertices, for *Why do we only define for*

Chart, line chart

Description automatically generatedFor example:

Chart

Description automatically generated with medium confidence

A picture containing sky

Description automatically generated

Q6. a) Draw the graph .

b) How many edges does the graph have?

c) How many non-isomorphic cycles are subgraphs of the graph below?

*List the vertices in the order you visit them for each possible cycle.*

Definition 9: A **Hamiltonian cycle** is a cycle that include every vertex in a graph.

Definition 10: Two vertices are **connected** if a path lies between them. A graph is **connected** if all its vertices are connected.

Example: In the graph below, an example of:

*It is ok to include a vertex or an edge more than once on a walk.*

* a **walk** is
* a **path** is
* a **trail** is

*It in not necessary to include every vertex in a cycle.*

* a **cycle** is
* a **Hamiltonian cycle** is

A picture containing wire

Description automatically generated

Q7. a) Draw a disconnected graph with 7 vertices.

1. Draw a graph with the minimum number of edges that contains both and as subgraphs.
2. Draw a connected graph which doesn’t contain a cycle.
3. Is it possible for a graph to contain a Hamiltonian cycle and a vertex of degree 1?
4. Is it ever possible to find a walk through a graph that is also a path, a trail, and a cycle?

Definition 11: A **complete graph** is a graph in which every vertex is directly connected to every other vertex with a single edge.

We can use the notation to represent the completed graph with vertices.

Shape, polygon

Description automatically generatedBelow are the graphs of and .

Shape, square, polygon

Description automatically generated with medium confidence

Q8. a) Draw the graphs and .

1. How many edges does the graph of have?
2. Draw all the possible subgraphs of , ignoring isomorphisms.

Definition 12: A **planar graph** is a graph that can be drawn in a plane without any edges meeting except at a vertex.

is a planar graph as shown by the two isomorphism below.

A picture containing shape

Description automatically generatedShape, square, polygon

Description automatically generated with medium confidence

*We can redraw where no edges cross over each other*

Radar chart

Description automatically generated with medium confidenceQ9. a) Draw an isomorphism of the graph below to show that it is planar.

isn’t planar.

1. Hence prove that also isn’t planar for .

The pigeonhole principle states that if you have pigeons and only pigeonholes then there must be at least one pigeonhole containing two or more pigeons.

More generally we can say if you have objects and containers, with , then at least one container must contain more than one item.

This seems pretty obvious, but it’s extremely useful in a lot of different mathematical proofs.

Q10. By considering a graph with vertices, make use of the pigeonhole principle to answer the following questions:

1. Show that every connected graph has two vertices of the same degree.
2. Show that this also holds for graphs which aren’t connected.

Definition 13: A **tree** is a connected graph which contains no cycles.

For example, the graph is a tree but the graph isn’t as it contains a .

Chart, line chart

Description automatically generatedChart, line chart

Description automatically generated

Definition 14: A **spanning tree** of a graph , is a subgraph which contains all the vertices in and is also a tree.

Chart, polygon

Description automatically generatedFor example, in the graph below a possible spanning tree is:

Shape, polygon

Description automatically generated

Q11. a) Find all possible trees containing 5 vertices, ignoring isomorphisms.

1. A picture containing wire, day

   Description automatically generatedFor the graph below, show that there isn’t a unique spanning tree.
2. How many edges will any spanning tree of graph with vertices have?

Q12. a) For any tree, explain why connecting any two vertices with a single edge will always create a cycle.

1. For any tree, explain why removing a single edge will always result in a disconnected graph.
2. A graph contains two vertices and . Prove that if there are two different paths connecting vertices and , then can’t be a tree.

Q13. (Research Question)

Find the definitions for each of the following graphs theory terms and explain them using your own words.

Draw an example to illustrate each term.

1. A leaf
2. A forest
3. A Eulerian circuit
4. A bipartite graph